Uniformly Bounded Superposition Operators in the Space of Second Bounded Variation Functions in the sense of Shiba

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Abstract

In this paper we introduce the notion of “function of second bounded variation” in the sense of Shiba and we show that if a superposition operator applies the space of all such functions on itself and it is uniformly bounded, then its generating function satisfies a Matkowski condition.

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1. Introduction

The notion of function of bounded variation, or \( BV \) function, was introduced by C. Jordan in 1881 \cite{1}, when he critically re-examined a faulty proof given by Dirichlet to the famous Fourier’s conjecture on trigonometric series expansion of periodic functions, see \cite{2}. By showing that functions of bounded variation are precisely those that can be expressed as the difference of two monotone functions, Jordan actually extended the Dirichlet’s criterium (on convergence of the Fourier series of monotone functions) to the class of \( BV \) functions. Since then, the notion has been generalized in several ways leading to many important results in mathematical analysis, among them, many concerning the nature of the composition of functions in those spaces.

If \( I = [a, b] \) is a closed interval in \( \mathbb{R} \), we will denote by \( BV(I, \mathbb{R}) \) the space of all (real valued) functions of bounded variation on the interval \( I \).
Inspired by the work done in [3], we will say that a sequence of positive real numbers, \( \Lambda = (\lambda_i)_{i=1}^{\infty} \), is a \( \mathcal{W} \)-sequence if it is non-decreasing and \( \sum (1/\lambda_i) = +\infty \).

Indeed, in 1972, D. Waterman [3] introduced the class \( \Lambda BV([a,b]) \) of functions of \( \Lambda \)-bounded variation and in 1980, M. Shiba [4] generalized Waterman’s notion by introducing the class \( \Lambda_p BV([a,b]) \) \((1 \leq p < \infty)\); that is, the class of all functions \( f : [a,b] \to \mathbb{R} \) with bounded \( \Lambda_p \)-variation on \([a,b]\).

**Definition 1.** [5] Let \( I = [a,b] \subset \mathbb{R} \) be a closed interval. Let \( \Lambda \) be a \( \mathcal{W} \)-sequence and suppose that \( p \geq 1 \). A function \( f : I \to \mathbb{R} \) is said to be of bounded \( \Lambda_p \)-variation on \( I \) if

\[
V_{\Lambda_p}(f) = V_{\Lambda}(f, p, I) = \sup_{\xi} V_{\Lambda}(\xi, f, p, I) < \infty,
\]

where

\[
V_{\Lambda}(\xi, f, p, I) := \left( \frac{1}{p} \sum_{i=1}^{n} \frac{|f(x_i) - f(x_{i-1})|^p}{\lambda_i} \right)^{1/p},
\]

and the supremum is taking over all partitions \( \xi : a = x_0 < x_1 < \cdots < x_n = b \) of the interval \( I \).

If \( p = 1 \) one obtains just the Waterman’s class \( \Lambda BV([a,b]) \).

We will use the expression \( V^l \) to denote the linear space of all functions from \( I \) into a given linear space \( V \). Let \( \mathcal{X} \) be a subspace of \( \mathbb{R}^l \). Given a function \( h : I \times \mathbb{R} \to \mathbb{R} \), the superposition (Nemytskij or composition, see [6]) operator \( H : \mathcal{X} \to \mathcal{Y} \), generated by \( h \), is defined as

\[
H(f)(t) := h(t, f(t)), \quad t \in [a,b].
\]

(1.1)

If the function \( h \) does not depend on the first variable; that is, \( h : \mathbb{R} \to \mathbb{R} \), then the operator

\[
H(f)(t) := h(f(t)), \quad t \in [a,b], \ f \in \mathcal{X}
\]

(1.2)

is known as the autonomous superposition operator generated by \( h \).

Given two linear spaces \( \mathcal{X}, \mathcal{Y} \subset \mathbb{R}^l \) and a function \( h : I \times \mathbb{R} \to \mathbb{R} \), a primary goal of research in nonlinear analysis is to investigate under what conditions, for the generating function \( h \), the associated superposition operator maps \( \mathcal{X} \) into \( \mathcal{Y} \). This problem is known as the Superposition Operator Problem; see, e.g., [6, 7]. Following [6] we will state the Superposition Operator Problem as the following set-theoretic identity:

\[
\text{sop} (\mathcal{X}, \mathcal{Y}) := \{ h : \text{sop}_h(\mathcal{X}) \subset \mathcal{Y} \}
\]

where \( \text{sop}_h(f)(t) := h(t, f(t)), \quad t \in [a,b] \) and \( f \in \mathcal{X} \), also we write just \( \text{sop}(\mathcal{X}) \) if \( \mathcal{X} = \mathcal{Y} \).

The first work on the SOP in the space \( BV(I, \mathbb{R}) \) was made by Josephy in 1981 [8]. In 1974, Chaïka and Waterman [9] reached a similar result for the space of functions of bounded harmonic variation \( HBV(I) \).

In a seminal article of 1982, J. Matkowski, [10], showed that if a superposition operator \( H \), generated by a function \( h : [a,b] \times \mathbb{R} \to \mathbb{R} \), maps the space \( \text{Lip}[a,b] \), of all Lipschitz functions defined on \([a,b]\), into itself and is globally Lipschitz, then \( h \) has the form

\[
h(x,y) = h_1(x)y + h_0(x), \quad x \in I, \ y \in \mathbb{R}
\]

(1.3)

for some \( h_1, h_0 \in \text{Lip}[a,b] \).

There are a variety of spaces besides \( \text{Lip}[a,b] \) that verify this result (see [11, 12]).

In this paper we introduce the notion of second bounded variation functions, in the sense of Waterman-Shiba, \( \Lambda_2 BV \), following the line traced by De la Vallée Poussin [13], in 1908, and M. Shiba [4], in 1980. Also, we will show that if a superposition operator applies the space \( \Lambda_2 BV \) in itself and it is uniformly bounded then its generating function must be of the form (1.3). In order to do that, in section 2 we present the definition, properties and some auxiliary results for the new space \( \Lambda_2 BV \).

2. The space \( \Lambda_2 BV \)

In these sections, we introduce the notion of functions of second bounded variation in the sense of Shiba.

Throughout the rest of this paper we will denote by \( V_{\Lambda_1,p}(f) \) or \( V_{\Lambda_1,p}(f; [a,b]) \) the \( \Lambda_2 \)-variation, given in the definition 1.
In addition, \( \pi = \{x_i\}_{i=0}^n \) will denote a partition of the interval \([a, b]\); that is, \( a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \). If such a partition includes at least three points we will write \( \pi \in \Pi_3([a, b]) \).

Suppose \( \Lambda = \{\lambda_i\}_{i=0}^\infty \) be a \('W'-sequence.

**Definition 2.** Let \( 1 \leq p < \infty \). The \((\Lambda, 2, p)\)-th variation of \( f \) on \([a, b]\) is defined as

\[
V_{\Lambda, 2, p}(f; [a, b]) = V_{\Lambda, 2, p}(f) = \sup_\pi \left( \sum_{i=0}^{n-2} \frac{|Q_i(f; x_{i+2}, x_{i+1}) - Q_i(f; x_{i+1}, x_i)|^p}{\lambda_i} \right)^{1/p},
\]

where \( Q_i(f; \beta, \alpha) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \) and the supremum is taken over all the partitions \( \pi = \{x_i\}_{i=0}^n \in \Pi_3([a, b]) \).

The sum in the definition 2, is called an approximate sum to \( V_{\Lambda, 2, p}(f; [a, b]) \).

When \( V_{\Lambda, 2, p}(f; [a, b]) < \infty \) we say that \( f \) has \((\Lambda, 2, p)\)-th bounded variation on \([a, b] \). We denote by \( \Lambda^2_{p}BV([a, b]) \) the space containing such functions.

**Lemma 1.** The space \( \Lambda^2_{p}BV([a, b]) \) contains all affine functions.

**Proof.** Let \( \Lambda \) be a \('W'-sequence and suppose \( \pi = \{x_i\}_{i=0}^n \in \Pi_3([a, b]) \).

Let us define \( \varphi(x) = cx + d \), \( x \in [a, b] \), \( c, d \in \mathbb{R} \). Then

\[
V_{\Lambda, 2, p}(\varphi; [a, b]) = 0 \quad \text{and so} \quad \varphi \in \Lambda^2_{p}BV([a, b]).
\]

\( \square \)

3. **Main Results**

This section is dedicated to establish some properties of the space \( \Lambda^2_{p}BV([a, b]) \). We start showing that \( \Lambda^2_{p}BV([a, b]) \) is a vector space.

**Lemma 2.** If \( 1 \leq p < \infty \), \( f, g \in \Lambda^2_{p}BV([a, b]) \), and \( \lambda \in \mathbb{R} \), then

\begin{itemize}
  \item \( V_{\Lambda, 2, p}(\lambda f) = |\lambda| V_{\Lambda, 2, p}(f) \) and
  \item \( V_{\Lambda, 2, p}(f + g) \leq V_{\Lambda, 2, p}(f) + V_{\Lambda, 2, p}(g) \).
\end{itemize}

**Proof.** Let \( \Lambda \) be a \('W'-sequence and and suppose \( \pi = \{x_i\}_{i=0}^n \in \Pi_3([a, b]) \). Then

\[
\left( \sum_{i=0}^{n-2} \frac{|Q_i(\lambda f; x_{i+2}, x_{i+1}) - Q_i(\lambda f; x_{i+1}, x_i)|^p}{\lambda_i} \right)^{1/p} = |\lambda| \left( \sum_{i=0}^{n-2} \frac{|Q_i(f; x_{i+2}, x_{i+1}) - Q_i(f; x_{i+1}, x_i)|^p}{\lambda_i} \right)^{1/p} = |\lambda| V_{\Lambda, 2, p}(f).
\]
So, \( V_{\Lambda,2,p}(\lambda f) = |\lambda| V_{\Lambda,2,p}(f) \). Moreover,

\[
\begin{aligned}
&= \left\{ \sum_{i=0}^{n-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{A_i} \right\}^{1/p} \\
&\leq \left\{ \sum_{i=0}^{n-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{A_i} \right\}^{1/p} \\
&+ \left\{ \sum_{j=m}^{n-2} \frac{|Q_1(f; x_{j+2}, x_{j+1}) - Q_1(f; x_{j+1}, x_j)|^p}{A_j} \right\}^{1/p}.
\end{aligned}
\]

Therefore, \( V_{\Lambda,2,p}(f + g) \leq V_{\Lambda,2,p}(f) + V_{\Lambda,2,p}(g) \).

Now, it is easy to prove that

\[ \|f\|_{\Lambda,2,p} := \|f\|_\infty + V_{\Lambda,2,p}(f; [a, b]) \]

defines a norm in \( \Lambda^2_{p,BV}([a, b]) \).

The following proposition shows that the variation \( V_{\Lambda,2,p}(f; [a, b]) \) is semi-additive.

**Theorem 1.** If \( a < c < b \), \( 1 \leq p < \infty \), and \( f \in \Lambda^2_{p,BV}([a, b]) \), then \( f \in \Lambda^2_{p,BV}([a, c]) \), \( f \in \Lambda^2_{p,BV}([c, b]) \) and

\[ (V_{\Lambda,2,p}(f; [a, b]))^p \geq (V_{\Lambda,2,p}(f; [a, c]))^p + (V_{\Lambda,2,p}(f; [c, b]))^p. \]

**Proof.** Consider \( c \in (a, b) \), and fix two partitions in \( \Pi_3([a, c]) \) and \( \Pi_3([c, b]) \) respectively; say

\( \pi_1: a = x_0 < x_1 < \cdots < x_{m-1} < x_m = c \)

and

\( \pi_2: c = x_m < x_{m+1} < \cdots < x_{n-1} < x_n = b. \)

Put

\[ A := \sum_{i=0}^{m-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{A_i}, \]

\[ B := \sum_{j=m}^{n-2} \frac{|Q_1(f; x_{j+2}, x_{j+1}) - Q_1(f; x_{j+1}, x_j)|^p}{A_j}, \]

and consider the partition \( \pi = \pi_1 \cup \pi_2 \) which is in \( \Pi_3([a, b]) \).

Then,

\[ A + B \leq A + B + \frac{|Q_1(f; x_{m+1}, x_m) - Q_1(f; x_m, x_{m-1})|^p}{A_{m-1}} \]

\[ = \sum_{i=0}^{n-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{A_i}, \]

thus

\[ (A + B)^{1/p} \leq \sup_{\pi \in \Pi_3([a, b])} \left\{ \sum_{i=0}^{n-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{A_i} \right\}^{1/p} \]

which implies \( (A + B) \leq (V_{\Lambda,2,p}(f; [a, b]))^p. \)

Therefore,
$A \leq (V_{A,2,p}(f; [a, b]))^p - B$

and taking supremum over all partitions $\pi_1$ of $[a, c]$ we obtain:

$$(V_{A,2,p}(f; [a, c]))^p = \sup_{\pi_1} A \leq (V_{A,2,p}(f; [a, b]))^p - B < \infty,$$

and so

$$f \in \Lambda_2^p BV([a, c]).$$

Hence,

$$B \leq (V_{A,2,p}(f; [a, b]))^p - (V_{A,2,p}(f; [a, c]))^p.$$

Now, taking supremum over all partitions $\pi_2$ of $[c, b]$ we have:

$$(V_{A,2,p}(f; [c, b]))^p = \sup_{\pi_2} B \leq (V_{A,2,p}(f; [a, b]))^p - (V_{A,2,p}(f; [a, c]))^p < \infty,$$

thus

$$f \in \Lambda_2^p BV([c, b])$$

and hence

$$(V_{A,2,p}(f; [a, c]))^p + (V_{A,2,p}(f; [c, b]))^p \leq (V_{A,2,p}(f; [a, b]))^p.$$

☐

**Lemma 3.** If $1 \leq p < \infty$ and $f \in \Lambda_2^p BV([a, b])$, then $Q_1(f; \cdot, \cdot)$ is bounded on $[a, b] \times [a, b]$.

**Proof.** We first note that if $a \leq x_0 < x_1 < x_2 < x_3 \leq b$, then, by the definition of $V_{A,2,p}(f; [a, b])$, it follows that:

$$|Q_1(f; x_0, x_1) - Q_1(f; x_1, x_0)|^p = |Q_1(f; x_0, x_2) - Q_1(f; x_2, x_1) + Q_1(f; x_2, x_1) - Q_1(f; x_1, x_0)|^p \leq 2^p |Q_1(f; x_0, x_2) - Q_1(f; x_2, x_1)|^p + |Q_1(f; x_2, x_1) - Q_1(f; x_1, x_0)|^p.$$

On the other hand, if $a < x_0 < x_1 < x_2 < x_3 \leq b$, let us take $\lambda_1$ and $\lambda_2$ with $\lambda_1 \leq \lambda_2$. Then

$$|Q_1(f; x_0, x_2) - Q_1(f; x_1, x_0)|^p \leq 2^p \left[ \lambda_2 |Q_1(f; x_0, x_2) - Q_1(f; x_2, x_1)|^p + \lambda_1 |Q_1(f; x_2, x_1) - Q_1(f; x_1, x_0)|^p \right] \leq 2^p \lambda_2 (V_{A,2,p}(f; [a, b]))^p.$$

Similarly, if $a = x_0 < x_1 < x_2 < x_3 \leq b$, let us take $\lambda_0$ and $\lambda_1$ with $\lambda_0 \leq \lambda_1$, so

$$|Q_1(f; x_0, x_2) - Q_1(f; x_1, x_0)|^p \leq 2^p \left[ \lambda_1 |Q_1(f; x_0, x_2) - Q_1(f; x_2, x_1)|^p + \lambda_0 |Q_1(f; x_2, x_1) - Q_1(f; x_1, x_0)|^p \right] \leq 2^p \lambda_1 (V_{A,2,p}(f; [a, b]))^p.$$

Now, let $c$ be an arbitrary point in $(a, b)$, $y_0, y_1 \in [a, b]$ and $A = |Q_1(f; c, a)|^p$.

The proof of the Lemma depend on how $y_0, y_1$ are located with respect to $a, b$ and $c$.  

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Case 1. Suponga que \( a < y_0 < c < y_1 < b \). Si \( y_2 \) es un punto tal que \( y_1 < y_2 < b \), entonces \( a < y_0 < c < y_1 < y_2 < b \) y esto implica que

\[
|Q_1(f; y_1, y_0)|^p = |Q_1(f; y_1, y_0) - Q_1(f; y_2, y_1) + Q_1(f; y_2, y_1) - Q_1(f; c, a) + Q_1(f; c, a)|^p
\]

\[
\leq 2^p|Q_1(f; y_1, y_0) - Q_1(f; y_2, y_1)|^p + 2^p|Q_1(f; y_2, y_1) - Q_1(f; c, a)|^p
\]

\[
+ 2^p|Q_1(f; c, a)|^p
\]

\[
= 2^p\lambda_1\left[|Q_1(f; y_2, y_1)|^p - |Q_1(f; y_1, y_0)|^p\right] + 2^p|Q_1(f; y_2, y_1)|^p + 2^p|Q_1(f; c, a)|^p
\]

\[
\leq 2^p\lambda_1(V_{A,\lambda,\rho}(f; [a, b]))^p + 2^p\lambda_1(V_{A,\lambda,\rho}(f; [a, b]))^p + \lambda_1 2^p \frac{A}{\lambda_1}.
\]

Por lo tanto,

\[
|Q_1(f; y_1, y_0)|^p < \lambda_1 \left[ (2^p + 2^p)(V_{A,\lambda,\rho}(f; [a, b]))^p + \frac{2^p A}{\lambda_1} \right] = K_1.
\]

Case 2. Suponga que \( a < y_0 < c < y_1 = b \). Si \( y_2 \) es un punto tal que \( c < y_2 < y_1 = b \), entonces \( a < y_0 < c < y_2 < y_1 = b \) y esto implica que

\[
|Q_1(f; y_1, y_0)|^p
\]

\[
= |Q_1(f; y_1, y_0) - Q_1(f; y_0, a) + Q_1(f; y_0, a) - Q_1(f; y_1, y_2)|^p
\]

\[
+ |Q_1(f; y_1, y_2) - Q_1(f; c, a) + Q_1(f; c, a)|^p
\]

\[
\leq 2^p|Q_1(f; y_1, y_0) - Q_1(f; y_0, a)|^p + 2^p|Q_1(f; y_0, a) - Q_1(f; y_1, y_2)|^p
\]

\[
+ 2^p|Q_1(f; y_1, y_2) - Q_1(f; c, a)|^p + 2^p|Q_1(f; c, a)|^p
\]

\[
\leq 2^p\lambda_0\left[|Q_1(f; y_1, y_0) - Q_1(f; y_0, a)|^p + 2^p\lambda_1 2^p(V_{A,\lambda,\rho}(f; [a, b]))^p
\]

\[
+ 2^p\lambda_1 2^p(V_{A,\lambda,\rho}(f; [a, b]))^p + \lambda_1 2^p \frac{A}{\lambda_1}
\]

\[
\leq 2^p\lambda_0(V_{A,\lambda,\rho}(f; [a, b]))^p + \lambda_1 2^p 2^p(V_{A,\lambda,\rho}(f; [a, b]))^p + \lambda_1 \frac{2^p A}{\lambda_1}
\]

\[
< \lambda_1 \left[ (2^p + 2^p)(V_{A,\lambda,\rho}(f; [a, b]))^p + \frac{2^p A}{\lambda_1} \right] = K_2.
\]

Case 3. Suponga que \( a < y_0 < y_1 \leq c < y_2 < b \). Si \( y_2 \) es un punto tal que \( c < y_2 < b \) entonces tenemos que \( a < y_0 < y_1 \leq c < y_2 < b \) y entonces

\[
|Q_1(f; y_1, y_0)|^p = |Q_1(f; y_1, y_0) - Q_1(f; b, y_2) + Q_1(f; b, y_2) - Q_1(f; c, a) + Q_1(f; c, a)|^p
\]

\[
\leq 2^p|Q_1(f; y_1, y_0) - Q_1(f; b, y_2)|^p + 2^p|Q_1(f; b, y_2) - Q_1(f; c, a)|^p
\]

\[
+ 2^p|Q_1(f; c, a)|^p
\]

\[
\leq 2^p\lambda_1\left[|Q_1(f; y_2, y_1)|^p - |Q_1(f; y_1, y_0)|^p\right] + 2^p|Q_1(f; y_2, y_1)|^p + 2^p|Q_1(f; c, a)|^p
\]

\[
\leq 2^p\lambda_2\left[|Q_1(f; y_2, y_1)|^p + 2^p|Q_1(f; y_2, y_1)|^p + \frac{2^p A}{\lambda_2}\right]
\]

\[
< \lambda_2 \left[ (2^p + 2^p)(V_{A,\lambda,\rho}(f; [a, b]))^p + \frac{2^p A}{\lambda_2} \right].
\]
Therefore

$$|Q_1(f; y_1, y_0)|^p < \lambda_2 \left(2^p + 2^{2p} + 2^{4p}(V_{\Lambda, 2, p}(f; [a, b]))^p + \frac{2^{4p}}{\lambda_2}A \right) = K_3.$$  

These cases are typical. The other 3 cases:

- $a = y_0 < c < y_1 < y_2 < b$,
- $a = y_0 < c < y_2 < y_1 = b$,
- $a < c \leq y_0 < y_1 < y_2 < b$,

it can be shown analogously that $Q_1(f; \cdot, \cdot)$ is bounded.

In the next result we prove that any $f \in \Lambda^2_p(BV([a, b]))$ is Lipschitz and therefore absolutely continuous on $[a, b]$.

Denote by $L_a^b(f)$ the Lipschitz constant of a function $f : [a, b] \to \mathbb{R}$; i.e.,

$$L_a^b(f) := \sup \left\{ \left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right| : x_1, x_2 \in [a, b], x_1 \neq x_2 \right\}.$$  

Lemma 4. If $f \in \Lambda^2_p(BV([a, b]))$, where $1 \leq p < \infty$, then $f$ is Lipschitz continuous on $[a, b]$.

Proof. By Lemma 3, there exist $K > 0$ such that

$$|Q_1(f; y, x)|^p \leq K, \quad \forall x, y \in [a, b],$$

thus,

$$|Q_1(f; y, x)| \leq K^{1/p}, \quad \forall x, y \in [a, b],$$

or equivalently:

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq K^{1/p}, \quad \forall x, y \in [a, b].$$

This means that $f \in Lip([a, b], \mathbb{R})$, with $L_a^b(f) \leq K^{1/p}$.  

4. Superposition Operators in $\Lambda^2_p(BV([a, b]))$

Below we present a characterization of the action of the composition operator (Nemytskij) on the space $\Lambda^2_p(BV([a, b]))$, showing that if a superposition operator $H$ applies this space into itself and it is uniformly bounded, then its generating function $h : [a, b] \times \mathbb{R} \to \mathbb{R}$, satisfies a, so called, Matkowski’s condition, that is,

$$h(x, y) = h_1(x)y + h_0(x), \quad x \in [a, b], \quad y \in \mathbb{R}$$

for some $h_1, h_0 \in \Lambda^2_p(BV([a, b]))$.

Theorem 2. Let $h : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a function such that, for any $x \in [a, b]$ fixed, the function $h(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous with respect to the second variable. If the superposition operator $H$, generated by $h$, acts on the space $\Lambda^2_p(BV([a, b]))$ and satisfies the inequality

$$\|H(\varphi) - H(\psi)\|_{\Lambda, 2, p} \leq \gamma(\|\varphi - \psi\|_{\Lambda, 2, p}), \quad \varphi, \psi \in \Lambda^2_p(BV([a, b])), \quad (4.1)$$

for some function $\gamma : [0, +\infty) \to [0, +\infty)$, then $H$ satisfies a Matkowski’s condition.

Proof. Fix $y_1, y_2 \in \mathbb{R}$, with $y_1 \neq y_2$. For $\alpha, \beta \in [a, b]$ such that $\alpha < \beta$, consider the function

$$\eta_{\alpha, \beta}(t) := \begin{cases} 0, & \text{if } a \leq t \leq \alpha \\ \frac{t - \alpha}{\beta - \alpha}, & \text{if } \alpha \leq t \leq \beta \\ 1, & \text{if } \beta \leq t \leq b \end{cases}$$
Let us define \( \varphi_j : [a, b] \to \mathbb{R}, \quad j = 1, 2 \) by:

\[
\varphi_j(t) = \frac{1}{2} \left[ \eta_{\alpha,j}(t)(y_1 - y_2) + y_j + y_2 \right], \quad j = 1, 2.
\]  

(4.2)

We note that

\[
\varphi_1(t) = \frac{1}{2} \left[ \eta_{\alpha,1}(t)(y_1 - y_2) + y_1 + y_2 \right] \quad \text{and} \quad \varphi_2(t) = \frac{1}{2} \left[ \eta_{\alpha,2}(t)(y_1 - y_2) + 2y_2 \right];
\]

thus

\[
\varphi_1(t) - \varphi_2(t) = \frac{y_1 - y_2}{2}
\]

and

\[
\|\varphi_1 - \varphi_2\|_{L^2} = \|\varphi_1 - \varphi_2\| \leq V_{\Lambda,2,p}(\varphi_1 - \varphi_2; [a, b]) = \frac{|y_1 - y_2|}{2}.
\]

By Lemma 1, \( \varphi_1, \varphi_2 \in \Lambda^2_{p}BV([a, b]) \). Now for any \( x, y \in [a, b] \) such that \( x < y \), taking \( \alpha = x \) and \( \beta = y \) in (4.2), we get:

\[
\varphi_1(x) = \varphi_2(y) = \frac{1}{2} [y_1 + y_2], \quad \varphi_2(x) = y_2 \quad \text{and} \quad \varphi_1(y) = y_1;
\]

thus,

\[
\frac{|[H(\varphi_1) - H(\varphi_2)](y) - [H(\varphi_1) - H(\varphi_2)](x)|}{y - x}
\]

\[
= \frac{|H(\varphi_1)(y) - H(\varphi_2)(y) - H(\varphi_1)(x) + H(\varphi_2)(x)|}{y - x}
\]

\[
= \frac{|h(y, \varphi_1(y)) - h(y, \varphi_2(y)) - h(x, \varphi_1(x)) + h(x, \varphi_2(x))|}{y - x}
\]

\[
= \frac{|h(y, y_1) - h(y, \frac{y_1 + y_2}{2}) - h(x, \frac{y_1 + y_2}{2}) + h(x, y_2)|}{y - x}.
\]

On the other hand,

\[
\frac{|[H(\varphi_1) - H(\varphi_2)](y) - [H(\varphi_1) - H(\varphi_2)](x)|}{y - x}
\]

\[
\leq L^b_{\alpha}(H(\varphi_1) - H(\varphi_2))
\]

and

\[
L^b_{\alpha}(H(\varphi_1) - H(\varphi_2)) \leq K^b = \left( C_1 [V_{\Lambda,2,p}(H(\varphi_1) - H(\varphi_2); [a, b])]^{p} + C_2 A \right)^{\frac{1}{p}}
\]

\[
\leq \left[ C_1 \|H(\varphi_1) - H(\varphi_2)\|_{L^2}^{p} + C_2 A \right]^{\frac{1}{p}}
\]

\[
\leq \left[ C_1 \|\varphi_1 - \varphi_2\|_{L^2}^{p} + C_2 A \right]^{\frac{1}{p}}
\]

\[
\leq C \left( \gamma \|\varphi_1 - \varphi_2\|_{L^2}^{p} + A \right)^{\frac{1}{p}},
\]

where \( K \) and \( A \) are given in the proof of Lemma 4 and 3 respectively, and \( C = \max\{C_1^{1/p}, C_2^{1/p}\} \) is a positive constant. Hence

\[
\frac{|h(y, y_1) - h(y, \frac{y_1 + y_2}{2}) - h(x, \frac{y_1 + y_2}{2}) + h(x, y_2)|}{y - x}
\]

\[
\leq C \left( \gamma \left( \frac{|y_1 - y_2|}{2} \right) \right)^{p} \left( \gamma \|\varphi_1 - \varphi_2\|_{L^2} + A \right)^{\frac{1}{p}} |y - x|.
\]

(4.3)

By hypothesis the operator \( H \) takes its values on the space \( \Lambda^2_{p}BV([a, b]) \), and since the constants functions are in that space, we have, for any \( y \in \mathbb{R} \), that
and therefore $h(\cdot, y)$ is continuous for all $y \in \mathbb{R}$. Taking limit when $y \to x$ in (4.3) we obtain:

\[
\left| h(x, y_1) - h(x, \frac{y_1 + y_2}{2}) - h(x, \frac{y_1 + y_2}{2}) + h(x, y_2) \right| = 0 \quad \forall x \in [a, b], \quad \forall y_1, y_2 \in \mathbb{R},
\]

thus,

\[
2h(x, \frac{y_1 + y_2}{2}) = h(x, y_1) + h(x, y_2), \quad \forall x \in [a, b].
\]

This shows that $h(x, \cdot)$ satisfies the equation of Jensen (see [14], pág. 351), and since $h(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, by Theorem 13.2.2 of [14] (pág. 354), exist $h_0, h_1 : [a, b] \to \mathbb{R}$ such that

\[
h(x, y) = h_1(x)y + h_0(x), \quad x \in [a, b], \quad y \in \mathbb{R}.
\]

Let us note that for each $y \in \mathbb{R}$, the function $h(\cdot, y) \in \Lambda^2_pBV([a, b])$, then using equation (4.4) we have, for $y = 0$ and $y = 1$, that

\[
h_0(x) = h(x, 0) \quad \text{and} \quad h_1(x) = h(x, 1) - h_0(x), \quad \forall x \in [a, b]
\]

therefore $h_0, h_1 \in \Lambda^2_pBV([a, b])$, which completes the proof.

\[\square\]

Definition 3. [15]

Let $\mathcal{Y}$ and $\mathcal{Z}$ be two metric (or normed) spaces. We say that $H : \mathcal{Y} \to \mathcal{Z}$ is uniformly bounded if, for all $t > 0$ there exists a nonnegative real number $\gamma(t) \geq 0$ such that for all set $B \subset \mathcal{Y}$

\[
\text{diam } B \leq t \Rightarrow \text{diam } H(B) \leq \gamma(t).
\]

Theorem 3. Suppose that $a, b \in \mathbb{R}$, with $a < b$, $n \in \mathbb{N}$ and that $h : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a function such that $\forall x \in [a, b]$, $h(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous in the second variable. If the superposition operator $H$, generated by $h$, acts on $\Lambda^2_pBV([a, b])$ and is uniformly bounded, then there exist functions $h_0, h_1 \in \Lambda^2_pBV([a, b])$ such that

\[
h(x, y) = h_1(x)y + h_0(x), \quad x \in [a, b], \quad y \in \mathbb{R},
\]

and

\[
H(\varphi)(x) = h_1(x) \varphi(x) + h_0(x), \quad \varphi \in \Lambda^2_pBV([a, b]), \quad (x \in [a, b]).
\]

Proof. Let $\varphi, \psi \in \Lambda^2_pBV([a, b])$.

If $t \geq 0$ is such that $||\varphi - \psi||_{\Lambda^2_2} \leq t$, then $\text{diam } \{\varphi, \psi\} \leq t$ and $\text{diam } H((\varphi, \psi)) \leq \gamma(t)$, since $H$ is uniformly bounded.

Taking $t = ||\varphi - \psi||_{\Lambda^2_2}$ we obtain

\[
\text{diam } H((\varphi, \psi)) \leq \gamma(t) = \gamma(||\varphi - \psi||_{\Lambda^2_2})
\]

which implies that

\[
||H(\varphi) - H(\psi)||_{\Lambda^2_2} = \text{diam } H((\varphi, \psi)) \leq \gamma(||\varphi - \psi||_{\Lambda^2_2}),
\]

and the result follows from Theorem 2.

\[\square\]

5. Referencias


